

Weak Chaos in a Quantum Kepler Problem.

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Abstract

Transition from regular to chaotic dynamics in a crystal made of singular scatterers $U(r) = \lambda|r|^{-\sigma}$ can be reached by varying either σ or λ . We map the problem to a localization problem, and find that in all space dimensions the transition occurs at $\sigma = 1$, i.e., Coulomb potential has marginal singularity. We study the critical line $\sigma = 1$ by means of a renormalization group technique, and describe universality classes of this new transition. An RG equation is written in the basis of states localized in momentum space. The RG flow evolves the distribution of coupling parameters to a universal stationary distribution. Analytic properties of the RG equation are similar to that of Boltzmann kinetic equation: the RG dynamics has integrals of motion and obeys an H -theorem. The RG results for $\sigma = 1$ are used to derive scaling laws for transport and to calculate critical exponents.

I. INTRODUCTION

In the theory of localization to which I. M. Lifshits made many seminal contributions [1], one is interested in the effect of random potential on the quantum-mechanical wavefunction. When the randomness is weak, the wavefunctions are extended throughout the whole system, whereas at sufficiently high disorder all wavefunctions become localized [2]. The transition between localized and extended states for many years has been at the center of interest in condensed matter physics [3]. Originally, the concept of localization was invented in order to understand properties of disordered metals and doped semiconductors. More recently, it became clear that the ideas and the language developed in the localization theory are useful in a much broader context. One of the areas of theoretical physics that has numerous links with the localization theory is Quantum Chaos. In this article we consider quantum chaos in a periodic Kepler problem, and solve this problem by using tools from the localization theory. The relation between the Quantum Chaos and Localization theories appears to be quite generic, and we believe that it will help to understand the transition from integrable to chaotic behavior in a broad class of quantum systems.

Traditionally, the field of Quantum Chaos deals with the relation between classical and quantum dynamics in chaotic systems. One is interested in the features of classical dynamics that are manifested in quantum dynamics, such as periodic orbits, ergodicity, etc. Another group of questions concerns the transition from integrable and chaotic quantum dynamics. Already in the classical case this transition appears to be a difficult problem, although

tractable within the KAM theory [4,5]. On the other hand, the transition in the quantum case, apparently, is not quite so well understood. For the reader's sake, below we review some basic facts.

It is known that classical Hamiltonian dynamics can be either integrable or chaotic. Integrability means that all variables can be separated, which leads to quasiperiodic motion, concentrated in a small part of the phase space. In contrast, the chaotic (nonintegrable) dynamics is ergodic, i.e., the system can be found arbitrarily close to a given point in the phase space.

One of the most challenging problems in the theory of classical dynamical systems is the transition from integrable to chaotic behavior when an integrable system is perturbed, and the perturbation violates the integrability. There exist two scenarios of this transition. In the case of a singular perturbation, even a perturbation of an arbitrarily small amplitude can drive the system into chaos. A smooth perturbation acts in a different way: according to the seminal KAM scenario (see [4,5]) the motion remains non-ergodic under a smooth perturbation if the latter is weak.

A good example of a singular classical Hamiltonian dynamics is the motion in a periodic lattice of Coulomb scatterers [6]. The dynamics in this problem can be mapped onto the problem of geodesics on a negative curvature compact space, which in the dynamical systems classification is known to be of the most unpredictable, and yet deterministic kind. The degree of ergodicity one has in this problem is expressed mathematically by so-called Bernoulli symbolic dynamics scheme. This dynamics leads to very fast mixing in the phase space quantified by Kolmogorov entropy. Another interesting example of a similar kind is the anisotropic Kepler problem [7]: a particle with anisotropic mass tensor moving in the Coulomb potential. It was shown by Gutzwiller that the existence of the trajectories that fall on the center breaks integrability at any potential strength. Qualitative features of the dynamics are similar to that of the periodic system of Coulomb scatterers, although the Bernoulli scheme is constructed in a different way [7].

In the quantum problem, a lot of interest recently was focused on the semiclassical theory of quantum billiards. The billiards represent one of the simplest problems in which one can study the relation between classical and quantum chaos (see review in [8]). The quantum problem of a periodic array of Coulomb scatterers was considered by Gutzwiller who developed a scattering mode representation [9] well suited for treating the semiclassical limit.

In this article we consider the integrability to chaos transition in *quantum* systems, focusing on the case of singular perturbation. The usual approach is to look for critical strength of the perturbation. Instead of that, however, we found it more natural and interesting to add another dimension to the problem, and to characterize the integrable to non-integrable transition as a function of the degree of singularity of the perturbation. It turns out that if the perturbation is both not strong and not singular enough the quantum behavior, namely the spectral statistics, will remain similar to the integrable one. On the other hand, when either the perturbation itself or its singularity is sufficiently strong, the behavior is chaotic. We also find an intermediate behavior for weak perturbations with the singularity taking certain critical value. This behavior can be called quasichaos, or *weak chaos*.

The problem we will be interested in is the motion of a quantum particle in a three-dimensional crystal of scatterers with Coulomb cores. It will be assumed that the particle

velocity is so high that the wavelength is much smaller than the crystal period. This problem describes transmission of fast charged particles through a crystal, assuming the particles energy is so high that scattering by the atomic core potential is important. (We ignore complications arising from inelastic scattering.) The relation of our problem to the real situation corresponding to experiments on channeling particles in crystals will be discussed at the end of the paper in Sec. IX.

Usually, for a particle moving in a periodic potential there is a notion of a Bloch state, i.e., the particle wavefunction can be represented as a superposition of few plane waves. However, when the scattering by the cores is strong the number of plane waves forming each Bloch state becomes very large, and the wavevectors are distributed roughly uniformly over all directions in the momentum space. This means that the particle is not moving along a straight line, but rather is diffusing. These two regimes, ballistic motion roughly along a straight line and diffusive motion with an irregular change of direction, correspond to the Bloch states *localized* or *delocalized* in momentum space.

The idea of our approach is the following. We treat the core potential as a perturbation, and write the problem using plane waves as a basis. In this basis, the main part of Hamiltonian, the kinetic energy, is diagonal. Since the scattering potential is periodic, the wavevectors of relevant plane waves form a lattice in momentum space. In the chosen basis we have a tight-binding localization problem: the potential matrix elements define hopping between different lattice sites, and the kinetic energy can be interpreted as on-site energy.

For a non-singular core potential one effectively has only short-range hopping in the localization problem in momentum space. In this case, according to the conventional theory of localization, all states remain localized if the potential strength is below some threshold. On the other hand, if the core potential is singular, its Fourier components decay slowly and one has to deal with a localization problem with long-range hopping. We will see that long-range hopping can delocalize the quantum states even at arbitrarily weak potential, if the potential singularity is sufficiently strong.

The states localized in momentum space correspond to quasiperiodic motion in real space, i.e., to almost integrable dynamics, whereas the states delocalized in the momentum space correspond to chaotic dynamics in real space. Generally, the transition from one regime to the other can be reached in two different ways: either by increasing the strength of the core potential, or by varying the power law of the potential singularity at the origin. Below we consider the situation when the potential is weak: $Ze^2/\hbar v < 1$, where v is the particle velocity, and Z is nuclear charge. In this case, naively, the scattering is weak and thus the Bloch states consist only of few plane waves. However, we will see below that because of the $1/r$ -singularity of the Coulomb potential the condition of weak scattering is more stringent: $Ze^2/\hbar v \ln pa_B/\hbar < 1$, where p is the particle momentum, and a_B is Bohr's radius of the cores. We will be interested in the situation when

$$Ze^2/\hbar v < 1, \quad \text{and} \quad Ze^2/\hbar v \ln(pa_B/\hbar) \gg 1, \quad (1)$$

In this case the scattering is strong, there is no ballistic motion, and the Bloch states consist of many plane waves.

However, in this case the motion is not ergodic because the number of plane waves forming the Bloch state is much less than the total number of plane waves with appropriate energy. The reason is, as we will see below, that this problem falls right on the critical line

of a localization transition. To study this transition we introduce a more general model with core potentials having an arbitrary power law singularity (see Sec. II), and consider the transition from localized to extended states in momentum space resulting from changing the power law. The interesting thing about this transition is that it occurs even in the weak coupling limit, whereas in the conventional Anderson localization problem with short-range hopping all states are localized at weak hopping. As a result, the nature of the transition is quite different from the conventional picture accepted for short-range hopping, which leads to a completely different critical behavior.

After introducing the model in Sec. II, and discussing the mapping to a localization problem in Sec. III, we proceed with solving the problem. It turns out that this problem is much more tractable than the conventional localization. The localization transition driven by long-range hopping has been studied previously and is well understood [12]. It is known that as the decay rate of hopping changes, one comes to a point at which there appears an infinite number of resonances, occurring at all length scales. The interaction of these resonances delocalizes the quantum states of the system. However, right at the transition point the density of resonances turns out to be low (provided that the coupling is weak), and one can construct a theory of the transition by treating the resonances as a kind of dilute gas. This makes it possible to construct a renormalization group theory (RG). We discuss the basics of the RG formalism in Sec. IV, identify the quantities characterizing the strength of coupling, and then in Sec. V derive an RG flow of the coupling parameters.

The RG equation derived in Sec. V has some interesting features. It has properties very similar to that of the Boltzmann equation, and can be studied by tools borrowed from statistical mechanics. In Sec. VI we find integrals of motion of the RG flow, derive an H -theorem, and obtain stationary distributions characterizing universal RG limit.

The similarity between our RG equation and the Boltzmann equation has the following physical meaning. The RG procedure involves subsequent diagonalization of resonances ordered in energies, which is the same as the ordering in increasing spatial scales, i.e., in the RG time $\xi = \ln R$. Because of weak coupling, the only important kind of resonances are pair resonances which are well separated in the RG time domain. This situation is analogous to the statistical mechanics of a gas, where because of low density only pair collisions are important. Because of long collision time, one can neglect the correlations of subsequent collisions. Thus, everything about gas dynamics can be extracted from the mechanics of a two-atom collision, which is the essence of the Boltzmann equation. Similarly, in our problem the role of elementary collision events is played by pair resonances. The RG equation accounts for the change of the coupling parameters resulting from one resonance, and treats the resonances as uncorrelated events occurring randomly in the RG time. This analogy is quite fruitful: one can define an “entropy” obeying an H -theorem, and show that the RG flow leads to some analog of statistical equilibrium, in which the distribution of couplings takes a universal limiting form (which is the analog of Maxwell distribution in a gas).

In Sec. VII we use the RG to study scaling properties of the states. For that, the RG is extended to include participation ratios of the states. We write down an RG equation for the flow of the participation ratio distribution, and by solving it determine the scaling exponent. The exponent turns out to scale with the coupling strength, $Ze^2/\hbar v$, and thus is small at weak coupling. This exponent has a geometric meaning of fractal dimension of the

space region occupied by an eigenstate.

In Sec. VIII we study transport in the momentum space, which in terms of Bloch states means angular diffusion. As it should be at the localization transition, the diffusion is of anomalous character, $t \sim \mathcal{D}^{-1}r^d$. We find that the “diffusion constant” \mathcal{D} scales with the coupling strength. The dimensionless conductivity estimated from the Einstein relation $\sigma = \mathcal{D}n$ is much less than unity. Scaling properties of the scale invariant density-density correlator are related to the scaling exponent for the participation ratios derived in Sec VII.

Finally, let us mention that the renormalization method employed in this work is not completely new. Similar ideas were used in the studies of one-dimensional quasiperiodic Schrödinger operators [10], and of localization in quasicrystals [11].

II. MODEL

To be specific, we will focus on the problem of a particle moving in a 3–dimensional periodic crystal of scatterers:

$$V(r) = \sum_{a \in L} U(r - a) , \quad (2)$$

where $U(r)$ is a potential of one scatterer, and L is a 3–dimensional lattice with periods a_1 , a_2 , and a_3 :

$$a_1 n_1 \mathbf{i} + a_2 n_2 \mathbf{j} + a_3 n_3 \mathbf{k} , \quad \text{where} \quad n_\alpha \in Z . \quad (3)$$

The interaction $U(r)$ has a power law singularity at small r , $U(r) \sim |r|^{-\sigma}$, and falls rapidly at distances much larger than the lattice spacings a_α , so that the sum (2) is well defined. For the model we take

$$U(r) = \frac{\lambda}{|r|^\sigma} e^{-\kappa|r|} \quad (4)$$

where the “effective radius” κ^{-1} of the interaction is chosen to be of the order of the distance between the scatterers: $\kappa a_\alpha \simeq 1$.

We are interested in the spectrum and eigenstates of the Schrödinger operator

$$-\frac{1}{2}\nabla^2 + V(r) \quad (5)$$

in the semiclassical range of energies:

$$E \gg \frac{\hbar^2}{ma_\alpha^2} \quad (6)$$

Eigenstates in the periodic potential are Bloch states $\psi_p(r)$ characterized by quasimomentum p in the Brillouin zone of the lattice L .

Physically, the model (2)–(6) corresponds to the problem of the scattering of a fast particle by a crystal. Most interesting is the case of charged particles, like electrons or muons, because it turns out that the Coulomb potential singularity, $U(r) \sim \frac{1}{r}$ lies on a

critical line of a transition from regular to chaotic dynamics. However, it is tutorial to consider the problem (2)–(6) with an arbitrary σ .

The problem (2)–(6) is closely related to so-called quantum billiards. In the case of billiards, typically, one considers a particle moving within a rectangular, rhombic, or square billiard with a scattering potential inside and a hard wall boundary condition on the walls. In such billiards all nontrivial dynamics is due to the scattering by the potential, since the dynamics in an empty billiard is integrable. The problem of a hard core disk potential is known as a Sinai billiard. It is natural to generalize it to potentials with a power law singularity (4).

The only difference between the billiard problem and our problem is in boundary conditions: hard wall for billiards vs. periodic in our case. It will be clear below that in both cases all results are the same. In fact, in our discussion of the mapping to a localization problem, as well as in the following treatment of localization, one can everywhere replace Bloch waves by standing waves without affecting any of the conclusions. Clearly, this is consistent with the mathematical equivalence of the billiard and the crystal problems that one has in the semiclassical limit (6).

III. MAPPING TO A LOCALIZATION PROBLEM

Let us write the problem in the basis of Bloch plane waves $e^{i(p+g)r}$, where p is the quasimomentum in the Brillouin zone, and \mathbf{g} is the vector of the lattice L' , reciprocal to L :

$$\mathbf{g} = \frac{2\pi}{a_1}n_1\mathbf{i} + \frac{2\pi}{a_2}n_2\mathbf{j} + \frac{2\pi}{a_3}n_3\mathbf{k}, \quad \text{where} \quad n_\alpha \in Z. \quad (7)$$

In this basis Schrödinger equation takes the form

$$Ec_{\mathbf{g}} = E_{\mathbf{g}}c_{\mathbf{g}} + \sum_{\mathbf{g}' \neq \mathbf{g}} U_{\mathbf{g}-\mathbf{g}'}c_{\mathbf{g}'}, \quad (8)$$

where

$$E_{\mathbf{g}} = \frac{(p+g)^2}{2m} + U_0, \quad (9)$$

and $U_{\mathbf{g}}$ are the potential Fourier components $U_{\mathbf{g}} = V^{-1} \int e^{i\mathbf{g}r} U(r) d^3r$, where $V = a_1 a_2 a_3$ is the volume of the lattice unit cell. For the interaction (4) one has

$$U_{\mathbf{g}} = \frac{4\pi\lambda}{|\mathbf{g}|^{3-\sigma}V} \frac{\Gamma(2-\sigma) \sin((2-\sigma) \tan^{-1} q/\kappa)}{(\kappa^2/g^2 + 1)^{(2-\sigma)/2}}, \quad (10)$$

which can be expanded at small and large \mathbf{g} , correspondingly:

$$U_{\mathbf{g}} = \begin{cases} A_\sigma |\mathbf{g}|^{-(3-\sigma)} V^{-1}, & |\mathbf{g}| \gg \kappa; \\ B_\sigma \kappa^{-(3-\sigma)} V^{-1}, & |\mathbf{g}| \ll \kappa, \end{cases} \quad (11)$$

where $A_\sigma = 4\pi\lambda\Gamma(2-\sigma) \sin(\pi\sigma/2)$, $B_\sigma = 4\pi\lambda \Gamma(3-\sigma)$.

It is natural to think of the problem (8) as an Anderson localization problem defined on the sites of the lattice L' , where $E_{\mathbf{g}}$'s are on-site energies, and $U_{\mathbf{g}-\mathbf{g}'}$ are hopping amplitudes.

Such an analogy is meaningful because the hopping is *effectively local*, since $U_{\mathbf{g}}$ falls at large $|\mathbf{g}|$. In the problem (8) we will find a delocalization transition at $\sigma = 1$ and any λ , no matter how weak.

Let us recall that in the conventional Anderson model with a hopping which is weak and short range, all states are localized. In our problem this is not so, because the hopping is actually long range: $U_{\mathbf{g}} \sim |\mathbf{g}|^{-(3-\sigma)}$ at $|\mathbf{g}| \gg \kappa$. In such a problem, even weak hopping delocalizes the states, provided the exponent in the hopping power law is less than the space dimension (see Sec. IV and Refs. [12,13]). We will argue that the delocalization occurs over a shell of sites that have almost equal energies. In the momentum space, such sites are close to a “Fermi sphere” which has dimension two (see Fig. X). Thus, at small λ we have two phases:

- (i) $3 - \sigma > 2$, localized states;
- (ii) $3 - \sigma < 2$, delocalized states.

Localization in momentum space ($\sigma < 1$) means that the wave function in real space is a linear combination of a small number of plane waves, or harmonics, i.e., it describes a quasiperiodic motion. On the other hand, the states delocalized in momentum space ($\sigma > 1$) correspond to a sum of many plane waves with almost equal wavelengths, but nearly isotropic distribution of wavevectors. In this case the dynamics is chaotic.

To see why effective space dimension equals two, let us replace $U_{\mathbf{g}-\mathbf{g}'}$ in (8) by an infinitely long range hopping with a constant amplitude

$$W \equiv \max_{\text{all } \mathbf{g}} \{U_{\mathbf{g}}\} = B_{\sigma} \kappa^{-(3-\sigma)} V^{-1} . \quad (12)$$

This problem corresponds to scattering by a δ -function potential, and to chaotic dynamics.

Upon making the hopping infinite range all hopping amplitudes increase, and therefore the number of sites \mathbf{g} over which the states delocalize should also increase. However, for the new problem the states can be easily written explicitly:

$$c_{\mathbf{g}}^{(i)} = \frac{WC^{(i)}}{E^{(i)} - E_{\mathbf{g}}} , \quad (13)$$

where $C^{(i)}$ is defined as the sum of all $c_{\mathbf{g}}$'s,

$$C^{(i)} = \sum_{\mathbf{g}} c_{\mathbf{g}}^{(i)} , \quad (14)$$

and the eigenenergy $E^{(i)}$ is found from the equation

$$\sum_{\mathbf{g}} \frac{W}{E^{(i)} - E_{\mathbf{g}}} = 1 . \quad (15)$$

From that, one finds that the distribution of momenta in each state is isotropic, and that the sites \mathbf{g} over which the i -th state will spread are all contained within the shell

$$E^{(i)} - \gamma W < E_{\mathbf{g}} < E^{(i)} + \gamma W , \quad (16)$$

where γ is of the order of one. Clearly, the quantity W just sets an upper bound on the delocalization shell in the problem (8).

We are interested in the situation when $W \ll E$, and hence the shell is very thin compared to its radius. Therefore, we can think of the shell as of an effective two dimensional “physical space” in which our localization problem is defined. The density of the on-site energies $E_{\mathbf{g}}$ can be calculated from the sphere $|p| = p_0 = \sqrt{2mE}$ surface area, $4\pi p_0^2$, and the free particle density of states $\nu(E) = mp_0 V / 2\pi^2 \hbar^3$. The density of the on-site energies, taken per unit area of the shell and per unit energy, is

$$n = V m p_0^{-1} (2\pi \hbar)^{-3} . \quad (17)$$

Another similarity with the localization problem is that the energies $E_{\mathbf{g}}$ can be treated as *quasirandom numbers*. To see this, one notes that $E_{\mathbf{g}}$ ’s are given, up to a constant, by squares of the lattice vectors lengths. However, in a generic lattice L all vectors have different lengths, except for those related by a sign reversal $\mathbf{g} \rightarrow -\mathbf{g}$. This implies different distances to the constant energy sphere $|p| = p_0$, and thus no occasional coincidence among $E_{\mathbf{g}}$ ’s. Thus, for now we assume the energies uncorrelated, and discuss consequences of their correlation later.

To summarize, we argued that the problem (8)–(10), from the point of view of the theory of localization, is equivalent to the problem

$$E c_{\mathbf{g}} = E_{\mathbf{g}} c_{\mathbf{g}} + \sum_{\mathbf{g}' \neq \mathbf{g}} U_{\mathbf{g}-\mathbf{g}'} c_{\mathbf{g}'} . \quad (18)$$

In the problem (18) the sites \mathbf{g} are uniformly distributed over a plane, and have uniform density $n = V m p_0^{-1} (2\pi \hbar)^{-3}$. It is assumed that in the semiclassical limit the number of lattice points around the constant energy shell becomes so large that one can replace the finite size sphere by an infinite plane without affecting localization properties. The nature of this approximation is similar to the one made in the conventional localization theory when one goes from a finite system of size much bigger than the localization radius to an infinite system.

The energies $E_{\mathbf{g}}$ in (18) are quasirandom numbers having unit spectral density, and the hopping amplitudes $U_{\mathbf{g}-\mathbf{g}'}$ are defined by

$$U_{\mathbf{g}-\mathbf{g}'} = \lambda_{new} |\mathbf{g} - \mathbf{g}'|^{-\alpha} , \quad (19)$$

where

$$\lambda_{new} = 4\pi \lambda_{old} V^{-1} \Gamma(2 - \sigma) \sin(\pi \sigma / 2) \quad \text{and} \quad \alpha = 3 - \sigma . \quad (20)$$

For convenience of notation, below we ignore the rescaling factor relating λ_{new} and λ_{old} , and denote λ_{new} by λ . This will not cause any ambiguity because from now λ_{old} will not appear.

Also, in the calculation done below we assume Coulomb potential for the scatterers, i.e., everywhere set $\sigma = 1$ and $\alpha = 2$. The generalization to other potentials will be discussed at the end of the paper, in Sec. X.

Finally, let us mention that it is possible to generalize our problem to other space dimensions. In fact, all arguments on the localization vs. delocalization behavior given above remain valid if one replaces the space dimension 3 by any d . In this case, the constant energy shell in momentum space, in which the localization problem is defined, has dimension $d - 1$. The Fourier transform of a power law potential $U(r) = \lambda r^{-\sigma}$ goes as $U_{\mathbf{g}} = \lambda |\mathbf{g}|^{d-\sigma}$.

Then, by the localization criterion which requires that the power law in the hopping spatial dependence equals the space dimension, we get that the transition occurs at σ satisfying the condition

$$\alpha \equiv d - \sigma = d - 1 . \quad (21)$$

This means that the Coulomb potential ($\sigma = 1$) has critical singularity in all space dimensions (obviously, with $d = 1$ excluded).

IV. DELOCALIZATION ARGUMENT: DIVERGING NUMBER OF RESONANCES

We take advantage of the mapping to the localization problem, and from now on, until Sec. IX, we deal with the localization problem with long range hopping. For brevity, we will use d to denote the space dimension in which localization takes place, which corresponds to $d - 1$ in the above sections. The hopping power law exponent is α (equal to $d_{\text{old}} - \sigma$). Let us recall that the character of localization depends on the nature of the hopping. For a short-range hopping the transition occurs at a particular ratio of the hopping to the on-site random potential. In contrast, for sufficiently long-range hopping, all the states are delocalized, no matter how weak the hopping is [2]. More precisely, in a d -dimensional systems with the $|\mathbf{g}|^{-\alpha}$ asymptotics of hopping localization can exist only for $\alpha > d$ (see [12]). If $\alpha < d$, then localization is destroyed, even at weak hopping. Thus, a localization-delocalization transition occurs at $\alpha = d$.

As a side remark, many real systems of interest do fall on the critical line $\alpha = d = 3$ (e.g., see [14]). Among such systems are localized optical phonons in disordered dielectric materials coupled by electric dipole forces, two-level systems in glasses coupled by an r^{-3} elastic interaction, magnetic impurities in metals coupled by the r^{-3} RKKY interaction, etc.

Let us briefly recall here the qualitative argument of Ref. [12] showing that the states of the problem (8)–(10) cannot be localized.

Consider two localized states located at \mathbf{g} and \mathbf{g}' having energies $E_{\mathbf{g}}$ and $E_{\mathbf{g}'}$. They are *resonance* if

$$|U_{\mathbf{g}-\mathbf{g}'}| \geq |E_{\mathbf{g}} - E_{\mathbf{g}'}| \quad (22)$$

If the resonance condition (22) holds, then eigenstates of the problem

$$\begin{aligned} E c_{\mathbf{g}} &= E_{\mathbf{g}} c_{\mathbf{g}} + U_{\mathbf{g}-\mathbf{g}'} c_{\mathbf{g}'} \\ E c_{\mathbf{g}'} &= E_{\mathbf{g}'} c_{\mathbf{g}'} + U_{\mathbf{g}'-\mathbf{g}} c_{\mathbf{g}} \end{aligned} \quad (23)$$

are not localized on \mathbf{g} or \mathbf{g}' , but are essentially non-zero at both sites, \mathbf{g} and \mathbf{g}' .

Now let us estimate the number of resonances and show that it diverges at $\alpha = d$. Let us take one site \mathbf{g} , and consider all sites \mathbf{g}' which fall in resonance with the site \mathbf{g} . The mean number of such sites is given by

$$\kappa_d \int |\mathbf{g}'|^{d-1} d|\mathbf{g}'| P[\mathbf{g}, \mathbf{g}'] , \quad (24)$$

where $P[\mathbf{g}, \mathbf{g}']$ is the probability that the condition (22) holds, and κ_d is the surface area of a d -dimensional unit sphere. To estimate $P[\mathbf{g}, \mathbf{g}']$, we note that since the on-site energies $E_{\mathbf{g}}$ are uniformly distributed with the density n , one simply has

$$P[\mathbf{g}, \mathbf{g}'] = |U_{\mathbf{g}-\mathbf{g}'}|n . \quad (25)$$

Substituting the probability (25) with $U_{\mathbf{g}-\mathbf{g}'} \sim \lambda|\mathbf{g}-\mathbf{g}'|^{-\alpha}$ into Eq. (24), one gets an estimate for the average number of resonances:

$$\lambda n \kappa_d \int |\mathbf{g}'|^{d-\alpha-1} d|\mathbf{g}'| . \quad (26)$$

This expression converges at large $|\mathbf{g}'|$ when $\alpha > d$, and diverges when $\alpha \leq d$. The divergence means that as one looks further away from a given site, there are more and more resonances. Obviously, in this case all the states must be delocalized.

It is peculiar that the localization criterion does not involve the strength of hopping, but only the decay rate at large distances. In the standard localization problem, with a short range hopping, the localization transition can be reached by varying the hopping strength. Whereas in our problem, starting from the localized state, the transition can be reached either by increasing the hopping strength or by decreasing α . This opens up a possibility to study the localization transition at hopping amplitudes much smaller than the on-site potential variance. In this regime, due to relative smallness of the hopping, one can develop an accurate theory of the transition (see Secs. V, VI).

To see more clearly what kind of simplification one has in the weak hopping limit, let us look again at Eq. (26) and estimate the number of resonances at critical $\alpha = d$. One has

$$\kappa_d n \int |\mathbf{g}'|^{-1} d|\mathbf{g}'| = \lambda n \kappa_d \ln(R_{\max}/R_{\min}) , \quad (27)$$

where R_{\max} is the size of the system, and R_{\min} is a “microscopic” scale. The number of resonances (27) increases logarithmically with the system size, which indicates delocalization. However, if the hopping strength λ is small, the effect of delocalization will be *weak*, because the resonances will occur rarely in the *log*-space. Another way to express it is to note that Eq. (27) means that the resonances have a uniform density over the *log*-shells:

$$2^k R_{\min} < |\mathbf{g} - \mathbf{g}'| < 2^{k+1} R_{\min} , \quad k = 1, 2, 3, \dots \quad (28)$$

This suggests treating the resonances “shell by shell,” i. e., a renormalization group approach. Since the density of the resonances distribution over the shells is small in λ one can construct an accurate and self-consistent RG scheme.

V. RENORMALIZATION GROUP THEORY.

In this section we review the RG theory of resonances [12]. The reason an RG treatment is required in this problem is that when all *direct* resonances discussed above are combined together and diagonalized, there appear subsequent resonances, basically of the same nature as primary resonances. Then, after the next generation of resonances is treated, there appear more resonances and this process must be repeated (as shown schematically in Fig. X).

To derive an RG equation an approximation will be made in which only pair resonances are taken into account, while all higher order resonances (triplets, quadruplets, etc.) are ignored. To motivate this approximation, let us discuss one property of resonances that will be basic for our approach.

Let two sites \mathbf{g} and \mathbf{g}' form a resonance. Consider another site \mathbf{g}'' which is in a resonance with any of these two. According to Eq. (27) we can estimate the probabilities

$$P\left(\frac{1}{2} \leq \frac{|\mathbf{g}' - \mathbf{g}|}{|\mathbf{g}'' - \mathbf{g}|} \leq 2\right) \simeq \lambda n \ll 1 \quad (29)$$

$$P\left(\frac{1}{2} \leq \frac{|\mathbf{g}'' - \mathbf{g}'|}{|\mathbf{g} - \mathbf{g}''|} \leq 2\right) \simeq \lambda n \ll 1 \quad (30)$$

(here 2 can be replaced by any other number of the order of one). In other words, if the three sites \mathbf{g} , \mathbf{g}' , and \mathbf{g}'' fall in a resonance, then one side of the triangle they form (say, $|\mathbf{g} - \mathbf{g}''|$) is much shorter than the other two ($|\mathbf{g} - \mathbf{g}'|$ and $|\mathbf{g}' - \mathbf{g}''|$). In accordance with Eq. (27), one has

$$\log_2 \left(\frac{\min(|\mathbf{g}'' - \mathbf{g}|, |\mathbf{g}'' - \mathbf{g}'|)}{|\mathbf{g} - \mathbf{g}'|} \right) \simeq (\lambda n)^{-1} \gg 1 \quad (31)$$

Therefore, one concludes that pair resonances are much more frequent than triplets: the estimate (29) shows that the probability of a triplet resonance is of order of λ^2 . Similar arguments show that the probabilities of finding resonances of k oscillators ($k = 4, 5, 6, \dots$) can be estimated as λ^{k-1} . This should be compared with the probability of a pair resonance which is of order of λ (see (29)). We see that pair resonances occur about λ^{-1} times more frequently than triplets, about λ^{-2} times more frequently than quadruplets, and so on. This justifies keeping only pairs in the RG.

The RG procedure involves the following steps. We truncate the $|\mathbf{g}|^{-d}$ -interaction at some R_0 , i.e., put $U_{\mathbf{g}-\mathbf{g}'} = 0$ for all pairs $(\mathbf{g}, \mathbf{g}')$ such that $|\mathbf{g} - \mathbf{g}'| > R_0$. Let us assume we know exact eigenstates for the truncated hamiltonian (call them R_0 -states). Then, to eliminate next renormalization shell, we replace R_0 by R_1 such that

$$R_1 \gg R_0, \quad \text{but} \quad \lambda n \log_2 \frac{R_1}{R_0} \ll 1 \quad (32)$$

Let us take the R_1 -states and represent them as linear combinations of the R_0 -states. According to the above discussion, all of R_1 -states are either single R_0 -states or resonance pairs of R_0 -states (we neglect by triple and all higher order resonances).

The consistency of the RG procedure relies on the fact that the localization radius of the R -states is less than R . One notes that by going from R_0 -states to R_1 -states, at weak hopping, the localization radius cannot exceed R_1 . In other words, on the scale R the states will look like a superposition of several states localized on a scale much smaller than R .

Practically, this means that at all R one can work in the basis of localized states. In the RG scheme, when considering the R_1 -states as a result of interaction (resonance) of the R_0 -states we assume that the resonating R_0 -states are *far apart* compared to their

localization radii. This enables to keep track of the coupling between the states, ignoring the details of the spatial structure of wavefunctions.

The RG flow preserves the form of the Hamiltonian, but modifies the hopping amplitudes. We will see below that the change of the amplitude $U_{\mathbf{g}-\mathbf{g}'}$ of hopping between \mathbf{g} and \mathbf{g}' is given by a multiplier that factors into $\bar{a}_{\mathbf{g}'}a_{\mathbf{g}}$. The quantities $a_{\mathbf{g}}$ have a meaning of coupling parameters generated by the RG flow. Mathematically, at each R the truncated Hamiltonian has the form

$$Ec_{\mathbf{g}} = E_{\mathbf{g}}c_{\mathbf{g}} + \sum_{|\mathbf{g}'-\mathbf{g}|<R} U_{\mathbf{g}-\mathbf{g}'}\bar{a}_{\mathbf{g}'}a_{\mathbf{g}} c_{\mathbf{g}'} , \quad (33)$$

where the quasirandom energies $E_{\mathbf{g}}$ and the hopping $U_{\mathbf{g}-\mathbf{g}'} = \lambda/|\mathbf{g}-\mathbf{g}'|^\alpha$ are the same as in the original problem (8)–(10), and the new quantities are the parameters $a_{\mathbf{g}}$ characterizing the strength of coupling between the states. Initially, all $a_{\mathbf{g}} = 1$. However, the RG procedure generates a non-trivial distribution of the parameters $a_{\mathbf{g}}$.

To see this in more detail, let us write down all necessary quantities for two states (at $R = R_0$) located at \mathbf{g} and \mathbf{g}' . They interact according to

$$\begin{aligned} Ec_{\mathbf{g}} &= E_{\mathbf{g}}c_{\mathbf{g}} + U_{\mathbf{g}-\mathbf{g}'}\bar{a}_{\mathbf{g}'}a_{\mathbf{g}} c_{\mathbf{g}'} , \\ Ec_{\mathbf{g}'} &= E_{\mathbf{g}'}c_{\mathbf{g}'} + U_{\mathbf{g}'-\mathbf{g}}\bar{a}_{\mathbf{g}}a_{\mathbf{g}'} c_{\mathbf{g}} , \end{aligned} \quad (34)$$

Two eigenstates c^+ and c^- are defined by

$$c^+ = \cos \theta c_{\mathbf{g}} + \sin \theta e^{i\phi} c_{\mathbf{g}'}, \quad c^- = -\sin \theta c_{\mathbf{g}} + \cos \theta e^{i\phi} c_{\mathbf{g}'} , \quad (35)$$

with

$$\tan \theta = \sqrt{\tau^2 + 1} - \tau , \quad \tau = \frac{E_{\mathbf{g}} - E_{\mathbf{g}'}}{2|U_{\mathbf{g}-\mathbf{g}'}\bar{a}_{\mathbf{g}'}a_{\mathbf{g}}|} , \quad \phi = \arg(\bar{a}_{\mathbf{g}'}a_{\mathbf{g}}) . \quad (36)$$

The energies of the states (36) are

$$E_{\pm} = \frac{1}{2} \left(E_{\mathbf{g}} + E_{\mathbf{g}'} \pm \sqrt{(E_{\mathbf{g}} - E_{\mathbf{g}'})^2 + 4|U_{\mathbf{g}-\mathbf{g}'}\bar{a}_{\mathbf{g}'}a_{\mathbf{g}}|^2} \right) \quad (37)$$

The transformation rule for the parameters $a_{\mathbf{g}}$ can be found from the relation

$$a_{\mathbf{g}}c_{\mathbf{g}} + a_{\mathbf{g}'}c_{\mathbf{g}'} = a^+c^+ + a^-c^- , \quad (38)$$

which means that

$$a^+ = \cos \theta a_{\mathbf{g}} + \sin \theta e^{-i\phi} a_{\mathbf{g}'} , \quad a^- = -\sin \theta a_{\mathbf{g}} + \cos \theta e^{-i\phi} a_{\mathbf{g}'} \quad (39)$$

The transformation rule (39) for the coupling parameters $a_{\mathbf{g}}$ shows that even when initially all $a_{\mathbf{g}} = 1$ the resonances lead to non-unit values of $a_{\mathbf{g}}$, which justifies including $a_{\mathbf{g}}$'s in the RG.

As a matter of fact, the parameters $a_{\mathbf{g}}$ are the only important ingredient of the RG. Other characteristics of the states, such as their energies, turn out to be irrelevant. The reason is that when a resonance of two states forms (say, at the scale R) the energies of the states split by something like $|U_{\mathbf{g}-\mathbf{g}'}| \sim R^{-\alpha}$. (Since at the scale R the resonances are typically spaced by

$|\mathbf{g} - \mathbf{g}'| \sim R$.) This splitting ensures that at larger scales these states never fall in resonance. Because of that, the energies $E_{\mathbf{g}}$ can be taken as uncorrelated quasirandom numbers with uniform distribution, and their change under RG transformations can be ignored.

Now, let us proceed with deriving an RG equation for the distribution of $a_{\mathbf{g}}$'s. In doing it we emphasize the analogy with the theory of Boltzmann kinetic equation. Boltzmann equation for gases is usually derived assuming the absence of correlations of subsequent collision processes, resulting from large mean free paths of molecules in a gas. Besides providing grounds for a probabilistic approach, the largeness of mean free path enables one not to take into account triple and other multiple collisions. Similarly, in our RG dynamics there is no repetition of resonances (collisions) of the same states, because of the energy splitting. Also, small density of the resonances in the \log -space, analogous to large mean free path in gases, makes triple and higher order resonances unimportant.

As a first step of deriving the RG equation, let us consider how the distribution of $a_{\mathbf{g}}$'s changes by going from R_0 -states to R_1 -states. The distribution $f(a)$ is defined by the probability density

$$dP = f(a)da d\bar{a} . \quad (40)$$

Under the change $R_0 \rightarrow R_1$ the distribution $f(a)$ changes to $\tilde{f}(a)$. The change is due to resonances ("collisions") occurring during the RG time interval $\ln R_0 < \xi < \ln R_1$. The difference $\tilde{f}(a) - f(a)$ can be obtained by integrating

$$\frac{1}{2} \left[\delta(a - a^+) + \delta(a - a^-) - \delta(a - a_1) - \delta(a - a_2) \right] \quad (41)$$

over

- 1) $f(a_1) da_1 d\bar{a}_1$;
- 2) $f(a_2) da_2 d\bar{a}_2$;
- 3) $nd^2\mathbf{g}$, where $\mathbf{g} = \mathbf{g}_1 - \mathbf{g}_2$ and n is the concentration of sites given by Eq. (17).
- 4) dE , where $E = E_1 - E_2$.

Here the subscripts "1", "2" label the states in a resonance pair. It is convenient to introduce a new variable τ instead of E , defined by the second equation in (36). The rotation angle θ is related with τ by $\tan \theta = \sqrt{\tau^2 + 1} - \tau$. The usefulness of the variable τ becomes clear from the identity

$$\begin{aligned} d^2\mathbf{g}dE &= 2\pi|\mathbf{g}|d|\mathbf{g}| 2|\bar{a}_1a_2U_{\mathbf{g}_1-\mathbf{g}_2}| d\tau = 4\pi|\mathbf{g}|^2|\bar{a}_1a_2U_{\mathbf{g}_1-\mathbf{g}_2}| |\mathbf{g}|^{-1}d|\mathbf{g}|d\tau \\ &= \left(4\pi|\mathbf{g}_1 - \mathbf{g}_2|^2|\bar{a}_1a_2U_{\mathbf{g}_1-\mathbf{g}_2}|\right) d(\ln |\mathbf{g}|) d\tau \end{aligned} \quad (42)$$

where the product $|\mathbf{g}_1 - \mathbf{g}_2|^2|\bar{a}_1a_2U_{\mathbf{g}_1-\mathbf{g}_2}|$ does not depend on $|\mathbf{g}_1 - \mathbf{g}_2|$. (Because $U_{\mathbf{g}_1-\mathbf{g}_2} = \lambda|\mathbf{g}_1-\mathbf{g}_2|^{-2}$.) Thus, we perform an integration over $\mathbf{g}_1-\mathbf{g}_2$ in the domain $R_0 < |\mathbf{g}_1-\mathbf{g}_2| < R_1$. The result is

$$\begin{aligned} \tilde{f}(a) - f(a) &= \ln(R_1/R_0) \lambda n \int d\tau d\bar{a}_1 da_1 d\bar{a}_2 da_2 |a_1| |a_2| f(a_1) f(a_2) \\ &\quad \times \left[\delta(a - a^+) + \delta(a - a^-) - \delta(a - a_1) - \delta(a - a_2) \right] \end{aligned} \quad (43)$$

According to the above discussion of the probability of pair resonances, the RHS in (43) is of order of $\lambda n \ln(R_1/R_0)$, which is a small number (see (32)). Then it is standard to replace Eq. (43) by a differential equation with respect to the RG time $\xi = \ln(R)$:

$$\begin{aligned} \frac{\partial}{\partial \xi} f(a) &= \lambda n \int d\tau d\bar{a}_1 da_1 d\bar{a}_2 da_2 |a_1| |a_2| \\ &\times f(a_1) f(a_2) \left[\delta(a - a^+) + \delta(a - a^-) - \delta(a - a_1) - \delta(a - a_2) \right]. \end{aligned} \quad (44)$$

This equation is the main result of this section. Our task now will be to look for solutions $f(a, \xi)$ to the problem (44) with the δ -function initial condition $f(a, \xi = 0) = \delta(a - 1)$ corresponding to the “microscopic” distribution of the parameters a_i .

VI. SOLVING THE RG EQUATION

The similarity with the Boltzmann equation makes the dynamics (44) very simple. The problem (44) has integrals of motion (analogous to the conservation of energy and momentum for Boltzmann equation), and also it obeys an H -theorem. Relying on these properties we analyze the asymptotical behavior of $f(a, \xi)$ at $\xi \rightarrow \infty$.

First, let us study the integrals of the dynamics (44). The simplest integral is

$$\langle |a|^2 \rangle = \int d\bar{a} da |a|^2 f(a, \xi) \quad (45)$$

Let us prove that

$$\frac{\partial}{\partial \xi} \langle |a|^2 \rangle = 0 \quad \text{or} \quad \int |a|^2 f(a, \xi) d\bar{a} da = \int |a|^2 f(a, 0) d\bar{a} da \quad (46)$$

Proof:

$$\frac{\partial}{\partial \xi} \langle |a|^2 \rangle = n \int d\tau \int d\bar{a}_1 da_1 d\bar{a}_2 da_2 f(a_1) f(a_2) \left[|a^+|^2 + |a^-|^2 - |a_1|^2 - |a_2|^2 \right] \quad (47)$$

But

$$|a^+|^2 + |a^-|^2 = |\cos \theta a_1 + \sin \theta e^{-i\phi} a_2|^2 + |-\sin \theta a_1 + \cos \theta e^{-i\phi} a_2|^2 = |a_1|^2 + |a_2|^2 \quad (48)$$

(unitary transformations preserve the norm). Consequently, the RHS of Eq. (47) vanishes. *QED*

Besides $\langle |a|^2 \rangle$ there exist other invariants of Eq. (44). All averages of the real and imaginary parts of $a = a' + ia''$,

$$\langle a'^2 \rangle, \quad \langle a''^2 \rangle, \quad \langle a' a'' \rangle, \quad (49)$$

are conserved by the same argument of orthogonal transformations preserving the norm.

An interesting issue to discuss is whether the conservation of the quantities (49) is an exact or an approximate result. One might worry that the conservation fails when higher order resonances are taken into account, in addition to the interacting pairs. However, the conservation laws remarkably survive. To see why that is, let us consider a resonance of k states. The Hamiltonian (8) is diagonalized by an orthogonal $k \times k$ matrix $U_{\alpha\beta}$, ($\alpha, \beta = 1, \dots, k$). One notes that the parameters a_α are transformed by the matrix U^{-1} , and then the conservation of the quantities (49) follows from orthogonality of U .

Another property of the problem (44) is an analog of the H -theorem. Let us define *entropy* of the distribution $f(a, \xi)$ as

$$H_\xi[f] = - \int \ln(|a|f(a, \xi)) f(a, \xi) d\bar{a}da \quad (50)$$

The H -theorem states that $H_\xi[f]$ grows monotoneously as function of the RG time ξ :

$$\frac{\partial}{\partial \xi} H_\xi[f] \geq 0 . \quad (51)$$

Proof: Using the identity $0 = \frac{\partial}{\partial \xi} 1 = \frac{\partial}{\partial \xi} \int f(a, \xi) d\bar{a}da = \int \frac{\partial}{\partial \xi} f(a, \xi) d\bar{a}da$ one finds

$$\begin{aligned} \frac{\partial H}{\partial \xi} &= - \int \ln(|a|f) \frac{\partial f}{\partial \xi} d\bar{a}da \\ &= n \int d\tau \int \int d\bar{a}_1 da_1 d\bar{a}_2 da_2 |a_1|f(a_1) |a_2|f(a_2) \ln \left[\frac{|a_1|f(a_1) |a_2|f(a_2)}{|a^+|f(a^+) |a^-|f(a^-)} \right] \end{aligned} \quad (52)$$

Another useful identity

$$\int d\tau \left[\int \int |a_1|f(a_1) |a_2|f(a_2) d\bar{a}_1 da_1 d\bar{a}_2 da_2 - \int \int |a^+|f(a^+) |a^-|f(a^-) d\bar{a}^+ da^+ d\bar{a}^- da^- \right] = 0 \quad (53)$$

follows from the fact that the variables a_1, a_2 and a^+, a^- are related by an orthogonal transformation. Combining the identity (53) with (52) one gets

$$\frac{\partial H}{\partial \xi} = n \int d\tau \int \int d\bar{a}_1 da_1 d\bar{a}_2 da_2 \Phi(a^+) \Phi(a^-) (X \ln(X) - X + 1) , \quad (54)$$

where $X = (\Phi(a_1) \Phi(a_2)) / (\Phi(a^+) \Phi(a^-))$, and $\Phi(u) = |u|f(u)$.

Finally, the quantity $X \ln(X) - X + 1$ is non-negative. From that, the H -theorem (51) follows momentarilly. *QED*

Note that although our entropy differs from Boltzmann entropy by $\langle \ln |a| \rangle$, it does not affect the proof of the H -theorem which follows the standard derivation (cf. Ref. [15]).

The problem (44) has *stationary distributions* given by

$$f_G(a) = \frac{A}{|a|} \exp(-a_\alpha G_{\alpha\beta} a_\beta) , \quad (55)$$

where $a_1 = a', a_2 = a''$, and \hat{G} is a positively defined symmetric 2×2 matrix.

Proof: If a solution $f(a, \xi)$ of Eq. (44) does not depend on ξ then $\frac{\partial H}{\partial \xi} = 0$ for it. Since $X \ln(X) - X + 1 = 0$ vanishes only for $X = 1$, according to Eq. (54) for such a solution $\Phi(a_1) \Phi(a_2) = \Phi(a^+) \Phi(a^-)$ for all θ (hence, for all τ). This is possible provided that

$$f(a) = f_G(a) . \quad (56)$$

Given $G_{\alpha\beta}$, the normalization constant A in (55) should be determined from the condition $\int f(a) d\bar{a}da = 1$. *QED*

Stationary solutions $f_G(a)$ are analogous to the Maxwell distribution which appears in statistical mechanics as a stationary solution of Boltzmann equation. An important distinction is that in the Maxwell distribution there is only one free parameter (temperature) while the distributions $f_G(a)$ are characterized by 3 parameters $G_{\alpha\beta}$ ($\alpha, \beta = 1, 2$; $\alpha \leq \beta$). Like in the Boltzmann equation theory, the number of parameters in the stationary distribution is equal to the number of conserved quantities (49).

Continuing the analogy with statistical mechanics, let us prove that our entropy is maximal on the stationary distributions (55). More precisely, let us take all functions $f(a)$ such that

$$\int f(a) d\bar{a}da = 1, \quad \int a_\alpha a_\beta f(a) d\bar{a}da = G_{\alpha\beta}, \quad (\alpha, \beta = 1, 2), \quad (57)$$

where $a_1 = a'$, $a_2 = a''$, $a = a' + ia''$. Then always $H[f] \leq H[f_G]$, where $f_G(a)$ is defined by (55); $H[f] = H[f_G]$ only if $f(a) = f_G(a)$.

Proof: Let us consider the first and the second variations of $H[f]$ constrained by (57) and by the normalization condition:

$$\begin{aligned} & \delta \left[H[f] - \lambda \int f(a) d\bar{a}da - \sum_{\alpha\beta} \mu_{\alpha\beta} \int a_\alpha a_\beta f(a) d\bar{a}da \right] \\ &= \int \left[-\ln(|a|f(a)) - 1 - \lambda - \sum_{\alpha\beta} \mu_{\alpha\beta} a_\alpha a_\beta \right] \delta f(a) d\bar{a}da \end{aligned} \quad (58)$$

$$\delta^2 \left[H[f] - \lambda \int f(a) d\bar{a}da - \sum_{\alpha\beta} \mu_{\alpha\beta} \int a_\alpha a_\beta f(a) d\bar{a}da \right] = - \int \frac{(\delta f(a))^2}{f(a)} d\bar{a}da < 0. \quad (59)$$

Since the second variation is negative, maximal value of $H[f]$ corresponds to

$$f(a) = \frac{1}{|a|} \exp \left(- (1 + \lambda) - \sum_{\alpha\beta} \mu_{\alpha\beta} a_\alpha a_\beta \right) \quad (60)$$

The relationship between $\lambda, \mu_{\alpha\beta}$ and $A, G_{\alpha\beta}$ is straightforward (compare (60) with (55)). *QED*

The asymptotic properties of solutions of Eq. (44) are thus rather simple: each solution converges to one of the stationary distributions $f_G(a)$. The parameters $G_{\alpha\beta}$ are completely determined by second moments of the initial distribution $f(a, \xi = 0)$:

$$G_{\alpha\beta} = \int a_\alpha a_\beta f(a, \xi = 0) d\bar{a}da \quad (\alpha, \beta = 1, 2, \alpha < \beta). \quad (61)$$

To conclude, the RG dynamics (44) has a 3-dimensional family of fix points parameterized by symmetric positively defined 2×2 matrices. The dynamics in the space of distributions $f(a)$ is shown schematically in Fig. X.

The parameters are determined by the initial distribution, which means that the coupling strength in this problem is not renormalized. Thus the critical behavior such as scaling of the states will depend on these parameters. This situation should be contrasted with

the ordinary localization transition at short-range hopping, where the critical behavior is believed to be completely universal, with no additional parameters.

Let us mention that for the particular problem of Coulomb scatterer we are interested in the distribution $f_G(a)$ has the form

$$f(a') = A|a'|^{-1} \exp(-Aa'^2) , \quad (62)$$

i.e., it does not depend on the imaginary part a'' . The reason is that in the initial distribution all parameters a are real: $f(a)|_{\xi=0} = \delta^{(2)}(a-1)$. The non-generic form of matrix $G_{\alpha\beta}$ resulting from the fact that all a 's are real leads to a complication: the stationary distribution (62) is not normalizable. However, since the divergence of $\int f(a')da'$ is only logarithmic (i.e., weak), the results in this case are the same as for generic $G_{\alpha\beta}$. The only subtlety is that the constant A in (62) is renormalized, at a rate proportional to $\ln \xi$, which leads to the distribution slowly flowing to more and more singular distributions. However, all results derive above still hold, including the absence of RG corrections to the coupling strength $\langle (a')^2 \rangle$. Because of that, the conclusion about non-universal RG fix point holds in this case as well.

VII. WAVEFUNCTION SCALING: RG FOR THE DISTRIBUTION OF PARTICIPATION RATIOS

In this section we study renormalization of the eigenstates. The hierarchical structure of the states (resonances of resonances of resonances ...) emerging from the RG dynamics suggests that they have fractal properties. To study the scaling behavior of the states we will consider their participation ratios. For a normalized state $c_{\mathbf{g}}^{(i)}$ its participation ratio $p^{(i)}$ is defined by

$$p^{(i)} = \sum_{\mathbf{g}} |c_{\mathbf{g}}^{(i)}|^4 \quad (63)$$

In the theory of localization participation ratios are used as a measure of the degree of localization. By studying scaling of participation ratios one can characterize fractal properties of the states.

Below we study how participation ratios are changed by the real space RG of the states described in Sec. V. For this purpose we need participation ratios $p_R^{(i)}$ of the R -states, which have been defined as true eigenstates of the system with the interaction truncated at the scale R . To incorporate the participation ratios in the RG, one can rewrite the RG formalism for the distribution $f(a, p)$, including both coupling strengths and participation ratios. In fact, the R -states are characterized not only by $a^{(i)}$ and $p^{(i)}$, but also by eigenenergies and positions. However, as we already discussed, the correlations between the energies and positions are not important, and thus at each step of the RG one can assume uniform uncorrelated distribution over energy and in space.

As it was argued above, the RG flow in the leading order is determined by pair resonances. Let us discuss how pair resonances affect the participation ratios, and derive an RG flow for the distribution $f(a, p)$. For that we consider the resonance pair (34) and find for it the participation ratios p^{\pm} of the states “+” and “−” as functions of p_i and p_j . Since the positions of the states i and j are well separated, one simply has:

$$p^+ = \cos^4 \theta p_i + \sin^4 \theta p_j, \quad p^- = \sin^4 \theta p_i + \cos^4 \theta p_j. \quad (64)$$

This change of the participation ratios must be considered together with the transformation of the parameters a_i, a_j given by Eq. (39). Other steps of the derivation of the RG flow for $f(a, p)$ are similar to those described in Sec. V, and the resulting RG equation is the following:

$$\begin{aligned} \frac{\partial}{\partial \xi} f(a, p) = & n \int d\tau d\bar{a}_1 da_1 d\bar{a}_2 da_2 dp_1 dp_2 |a_1| |a_2| f(a_1, p_1) f(a_2, p_2) \\ & \times [\delta(a - a^+) \delta(p - p^+) + \delta(a - a^-) \delta(p - p^-) \\ & - \delta(a - a_1) \delta(p - p_1) - \delta(a - a_2) \delta(p - p_2)] . \end{aligned} \quad (65)$$

Of course, from the flow equation (65) one can return to Eq. (44) by using the relation between the functions $f(a)$ and $f(a, p)$:

$$f(a) = \int_0^\infty f(a, p) dp \quad (66)$$

By using the RG flow for the distribution $f(a, p)$ given by Eq. (65) one can study the ξ -dependence of various moments of p and of a , such as $\langle p^k \rangle$ or $\langle |a|^m \rangle$, as well as correlators $\langle p^k |a|^m \rangle$. Here we concentrate only on the distribution of the participation ratios, assuming the distribution of the parameters a to be stationary. In other words, we will be interested only in the scaling limit of the problem.

Let us focus on calculating the average participation ratio

$$\langle p \rangle = \int \int \int d\bar{a} da dp p f(a, p) |_{\xi=\ln R} . \quad (67)$$

To facilitate the analysis we introduce a new function $s(a)$ defined by

$$s(a) = \int_0^\infty p f(a, p) dp . \quad (68)$$

The RG flow for $s(a)$ can be obtained by integrating Eq. (65) over p :

$$\begin{aligned} \frac{\partial}{\partial \xi} s(a) = & n \int d\tau d\bar{a}_1 da_1 d\bar{a}_2 da_2 dp_1 dp_2 |a_1| |a_2| f(a_1, p_1) f(a_2, p_2) \\ & \times [p^+ \delta(a - a^+) + p^- \delta(a - a^-) - p_1 \delta(a - a_1) - p_2 \delta(a - a_2)] . \end{aligned} \quad (69)$$

The integrals over p_1 and p_2 in Eq. (69) can be calculated by using (68), (64) and (66):

$$\frac{\partial}{\partial \xi} s(a) = n \int d\tau d\bar{a}_1 da_1 d\bar{a}_2 da_2 |a_1| |a_2| [s(a_1) f(a_2) [\cdot]_1 + s(a_2) f(a_1) [\cdot]_2] , \quad (70)$$

where

$$[\cdot]_1 = \cos^4 \theta \delta(a - a^+) + \sin^4 \theta \delta(a - a^-) - \delta(a - a_1) \quad (71)$$

and

$$[\cdot]_2 = \sin^4 \theta \delta(a - a^+) + \cos^4 \theta \delta(a - a^-) - \delta(a - a_2) \quad (72)$$

In the scaling limit $\xi \rightarrow \infty$, as we demonstrated above, the distribution $f(a)$ converges at large ξ to a stationary distribution $f_G(a)$. Since we are interested in the asymptotical behavior of $s(a)$ at $(\xi \rightarrow \infty)$, the function $f(a)$ in Eq. (70) can be set equal to the stationary distribution $f_G(a)$ given by Eq. (55). After that we get a *linear* integral equation for $s(a)$ with a kernel independent of ξ .

With that in mind, one can view Eq. (70) as

$$\frac{\partial}{\partial \xi} s(a) = -\hat{B}(s(a)) , \quad (73)$$

where \hat{B} is a constant (ξ -independent) linear operator which acts in the space of functions $s(a)$. Asymptotical behavior of $s(a)$ at $\xi \rightarrow \infty$ is then given by

$$s(a, \xi) \simeq \exp(-\mu \xi) s_0(a) , \quad (74)$$

where μ is the *lowest eigenvalue* of \hat{B} and $s_0(a)$ is the corresponding eigenvector: $\hat{B}(s_0(a)) = \mu s_0(a)$. Then, according to Eq. (67), the scaling behavior of $\langle p \rangle$ is given by

$$\langle p \rangle = \int d\bar{a} d a s(a) |_{\xi=\ln R} \simeq R^{-\mu} . \quad (75)$$

Therefore, the scaling exponent of $\langle p \rangle$ is related to μ , the lowest eigenvalue of \hat{B} .

In principle, if the stationary distribution $f_G(a)$ is known, μ can be determined (perhaps, numerically). However, here we are interested in estimating μ only by the order of magnitude.

To estimate μ , let us note that the kernel in Eq. (70) is of the order of

$$n \langle |a_1 a_2| \rangle \simeq n \langle a^2 \rangle = \lambda n . \quad (76)$$

Hence, given n and λ ,

$$\mu = \gamma \lambda n , \quad (77)$$

where γ is a constant of the order of one.

It is tutorial to derive the result (77) by another method. Using Eq. (70) one can write

$$\begin{aligned} -\mu &= \frac{\partial}{\partial \xi} \ln(\langle p \rangle) |_{\xi \rightarrow \infty} = \langle p \rangle^{-1} \int \frac{\partial}{\partial \xi} s(a) d\bar{a} d a \\ &= n \langle p \rangle^{-1} \int d\tau d\bar{a}_1 d a_1 d\bar{a}_2 d a_2 |a_1| |a_2| \left[s(a_1) f_G(a_2) [\cdot]_1 + s(a_2) f_G(a_1) [\cdot]_2 \right] , \end{aligned} \quad (78)$$

where

$$[\cdot]_1 = [\cdot]_2 = \sin^4 \theta + \cos^4 \theta - 1 . \quad (79)$$

But $\sin^4 \theta + \cos^4 \theta - 1 = -2 \sin^2 \theta \cos^2 \theta < 0$, and hence Eq. (78) can be rewritten as:

$$-\mu = -4n \int d\tau \sin^2 \theta \cos^2 \theta \int d\bar{a} d a f_G(a) \left(\int d\bar{a} d a |a| s(a) / \left(\int d\bar{a} d a s(a) \right) \right) \quad (80)$$

By estimating the RHS of Eq. (80) by the order of magnitude, one again obtains the result (77) and in addition proves that μ is positive.

The exponent μ has a simple geometric meaning. For a typical R -state, it gives the fractal dimension of a region occupied by this state. Hence, we find that in this problem the fractal dimension of the states depends on the coupling strength λn .

The argument presented above for the mean participation ratio, after proper modification, is also valid for any moment of the participation ratios. Hence, all other fractal dimensions must scale with λn the same way as $\langle p \rangle$.

In the localization problem it is sometimes of interest to consider the distribution of all moments of the wavefunction:

$$p(m) = \sum_{\mathbf{g}} |c_{\mathbf{g}}^{(i)}|^m. \quad (81)$$

The participation ratio studied above corresponds to $m = 4$. The distribution of $p(m)$ can be used to study fractal dimensions of the states.

It is straightforward to generalize our RG formalism for the quantities $p(m)$. For that, one simply notes that for the pair resonance the change of $p(m)$'s is written as

$$p^+(m) = |\cos \theta|^m p_i(m) + |\sin \theta|^m p_j(m), \quad p^-(m) = |\sin \theta|^m p_i(m) + |\cos \theta|^m p_j(m). \quad (82)$$

Putting it together with the transformation (39) leads to the RG flow for the distribution $f(p(m), a)$ of the same form as Eq. (65), but with p^\pm given by Eq. (82). The analysis of this RG equation is similar to that presented above, and so are the results. The scaling exponents for the quantities $p(m)$ are of the order λn . This confirms our conclusion about fractal dimension of the space region occupied by an eigenstate being of the order of λn .

VIII. TRANSPORT: ANOMALOUS DIFFUSION AND SCALING

In this section we study time-dependent transport. Particularly we are interested in the density correlation function

$$K(\mathbf{g} - \mathbf{g}', t) = \sum_{i,j} \langle e^{-i(E_i - E_j)t} \bar{c}_{\mathbf{g}}^{(i)} c_{\mathbf{g}}^{(j)} \bar{c}_{\mathbf{g}'}^{(j)} c_{\mathbf{g}'}^{(i)} \rangle, \quad (83)$$

where $c_{\mathbf{g}}^{(i)}$ are eigenstates, E_i are their energies, and $\langle \dots \rangle$ stands for averaging over the on-site disorder.

We will argue that the dynamics is critical (anomalous diffusion) and write down a scale invariant expression (89) for the correlation function $K(g, g', t)$. Then, by using the RG for the participation ratios presented in Sec. VII we will derive scaling properties of the function K .

Let us begin with a qualitative picture of propagation of an excitation in the system. Let the particle start at $t = 0$ from one of the sites (say, from the site 1 in Fig. X). After some time T it hops to its nearest resonance neighbor, site 2. The hopping time T is of the order of inverse resonance splitting, which, according to Sec. IV, can be estimated as

$$T \sim |E_+ - E_-|^{-1} \approx |\mathbf{r}_1 - \mathbf{r}_2|^2 / \lambda . \quad (84)$$

After the time of the order of T the particle is localized not only at the site \mathbf{r}_1 , but both at \mathbf{r}_1 and at \mathbf{r}_2 , since both modes c^+ and c^- are excited.

Later, after some larger time $T' \gg T$ new site will be involved in transport. Each of the two excited states sets in motion other states which themselves can be considered as a result of interaction of several states during the time interval $\leq T'$. This is shown in Fig. X, where new resonance states participating in the transport are linear combinations of c_3, c_4 and of c_5, c_6 , respectively. Obviously, $L' = |\mathbf{r}_1 - \mathbf{r}_\alpha| \simeq |\mathbf{r}_2 - \mathbf{r}_\alpha| \gg |\mathbf{r}_1 - \mathbf{r}_2|$, where $\alpha = 3, 4, 5, 6$. Thus, at times t which are longer than T' but shorter than the time necessary for exciting new states, the excitation is mainly carried by four states. According to the estimates made in Sec. IV, one has

$$L'/L \simeq |\mathbf{r}_1 - \mathbf{r}_\alpha|/|\mathbf{r}_1 - \mathbf{r}_2| \simeq \exp(\text{const}/(\lambda n)) , \quad T'/T \simeq (L'/L)^2 , \quad (85)$$

where $\alpha = 3, 4, 5, 6$.

Even later, after some time T'' ($T'' \gg T'$) new states are involved in transport. The corresponding sites are separated from the initial site by a large distance L'' such that

$$L''/L' \simeq \exp(\text{const} \cdot \lambda^{-1}) , \quad T''/T' \simeq (L''/L')^2 . \quad (86)$$

At this stage, the number of modes carrying the excitation again approximately doubles.

Let us remark here that the excitation evolution described above should be understood in a statistical sense. For a particular state, there is no exact length scale at which the number of resonances doubles. As we saw in Sec. IV, the hierarchy of resonances is characterized by density uniform in the log of spatial scale. The same is true for the time hierarchy of resonances.

To summarize, the larger the time, the more states are excited. Asymptotically, after a large time T about

$$2^{c\lambda n \log(T)} = T^{c\lambda n} \quad (87)$$

states are involved in the transport (here c is a number of the order of one to be determined later).

There are two conclusions one can draw. First, the RG picture of the states hierarchy in space is in a one-to-one correspondence with the hierarchy in the time domain. The sequence of times at which new states are involved in dynamics is determined by the spatial hierarchy of resonances.

Second, according to the above discussion, after a long time T the excitation spreads over a region of size $L(T) \simeq T^{1/2}$. This means that the dynamics is (asymptotically) *invariant* under rescaling

$$(\mathbf{r}, t) \rightarrow (Z\mathbf{r}, Z^2 t) , \quad (88)$$

where the rescaling factor Z is an arbitrary number.

In particular, the correlation function $K(\mathbf{r}, t)$ must be invariant under the rescaling (88). Thus we have

$$K(\mathbf{r}, t) = t^{-1} F(t^{-1/2} \mathbf{r}) , \quad (89)$$

at large t , $|\mathbf{r}|$. The function $F(\mathbf{x})$ is normalized,

$$\int F(\mathbf{x}) d^3 \mathbf{x} = 1 , \quad (90)$$

due to probability conservation. The particular form of $F(\mathbf{x})$ depends in some universal way on the distribution of couplings $f(a)$.

Let us note here that the simple scaling relation (89) is a direct consequence of the existence of stationary fix points $f_G(a)$ of the RG flow for $f(a)$ given by Eq. (44). Indeed, the coupling of complex modes composed of many single-site states is determined by the distribution of the parameters a_i given by the limiting distribution $f_G(a) = \lim_{\xi \rightarrow \infty} f(a, \xi)$. Because this limit exists, the coupling of complex R -states asymptotically scales as R^{-2} . The time of hopping scales as inverse coupling, which leads to the space-time scaling relation $T \sim R^2$.

There is a relation between the behavior of the scaling function $F(\mathbf{x})$ and small \mathbf{x} and the participation ration scaling exponent μ discussed in Sec. VIII (see Eqs. (75), (77)). It turns out that

$$F(\mathbf{x}) = |\mathbf{x}|^{-(2-\mu)} \quad \text{at} \quad |\mathbf{x}| \ll 1 . \quad (91)$$

To prove this relation, let us consider a finite system of size R and take the density correlator K given by (83) at times $t \leq R^2$. By the order of magnitude, K can be estimated by removing from Eq. (83) all terms with non-equal energies $E_i \neq E_j$. (This is correct because typically $|E_i - E_j|$ is larger than level spacing, which is of the order R^{-2} , and hence the phase factors $(E_i - E_j)t$ in (83) are large, which leads to random relative signs of all terms with $i \neq j$, and leads to mutual cancellations.)

On the other hand, after the terms with $i \neq j$ are removed, one gets the following estimate:

$$K(\mathbf{g}, \mathbf{g}', t \simeq R^2) \simeq \sum_i |c_{\mathbf{g}}^{(i)}|^2 |c_{\mathbf{g}'}^{(i)}|^2 . \quad (92)$$

Now, let us set $\mathbf{g}' = \mathbf{g}$ and sum over all \mathbf{g} :

$$\sum_{\mathbf{g}'=\mathbf{g}} K(\mathbf{g}, \mathbf{g}', t \simeq R^2) = \sum_{i, \mathbf{g}} |c_{\mathbf{g}}^{(i)}|^4 = \sum_i p^{(i)} , \quad (93)$$

where $p^{(i)}$ are the participation ratios. The right hand side of Eq. (93) is of the order $R^{2-\mu}$ (see Eqs. (75), (77)). This is consistent with the scaling form

$$K(\mathbf{g}, \mathbf{g}', t) = t^{-1} F(|\mathbf{g} - \mathbf{g}'|/t^{1/2}) \quad (94)$$

only if the function F satisfies Eq. (91).

Also, let us remark that in an arbitrary space dimension d the critical law of hopping, $U_{\mathbf{g}-\mathbf{g}'} \sim |\mathbf{g} - \mathbf{g}'|^{-d}$, leads to anomalous diffusion: $T \sim R^d$. The reason is that, as it is clear from the above discussion, the scaling transformation (88) changes to

$$(\mathbf{r}, t) \rightarrow (Z\mathbf{r}, Z^d t) , \quad (95)$$

and the scale-invariant function $F(\mathbf{x})$ is now defined by

$$K(\mathbf{r}, t) = t^{-1} F(t^{-1/d} \mathbf{r}) . \quad (96)$$

($\int F(\mathbf{x}) d^d \mathbf{x} = 1$). The anomalous diffusion follows directly from (96):

$$\langle |\mathbf{r}|^2 \rangle = \int |\mathbf{r}|^2 K(\mathbf{r}, t) d^d \mathbf{r} \approx t^{2/d} \int |\mathbf{x}|^2 F(\mathbf{x}) d^d \mathbf{x} . \quad (97)$$

In the problem of three-dimensional Coulomb scatterers, the space in which we have a localization problem is two-dimensional (constant energy surface in momentum space), and thus in this case there is no distinction between anomalous diffusion and ordinary diffusion.

To compare the results with the standard localization theory assuming short-range hopping, let us point out that diffusion constant in our problem is of the order λ . Dimensionless conductivity can be found from the Einstein relation $\sigma = n\mathcal{D}$ to be of the order $\lambda n \ll 1$. Thus in this problem one has critical behavior with $\sigma \ll 1$. On the other hand, in the delocalized phase of the system with short-range hopping one always has $\sigma \leq 1$.

IX. SCATTERING OF CHARGED PARTICLES IN CRYSTALS

Let us return to the problem of charged particle scattering in a crystal, and consider the dynamics of a fast charged particle (electron or muon) in a real crystal. Since atoms cores are just $1/r$ -scatterers, one expects that the effects of weak chaos we discussed above will take place, and lead to delocalization in momentum space, i. e., to the absence of Bloch waves. Of course, from a practical view point, there is always some inelastic scattering, which complicates observing the effects we are interested in. However, in the following we ignore such processes, as well as all possible effects of crystal boundaries.

Let us estimate the dimensionless coupling constant λn . From (17) and (11) one has

$$\lambda n = \frac{4\pi Z e^2}{\hbar v} , \quad (98)$$

where Z is the nucleus charge and v is the particle velocity. The coupling constant is of the order of one when the velocity v is of an atomic scale.

From (98) it is clear that with electrons one never has a strong coupling situation for a semiclassical problem. The reason is that if an electron is moving with velocity of an atomic scale, $v = Ze^2/\hbar$, or slower, the number of wavelengths in the region where the core potential is not screened is of the order of one. On the other hand, if the velocity increases so that the wavelength becomes small, one has a well defined, but not a very interesting semiclassical problem, because the coupling (98) becomes very weak. Therefore, channeling of fast electrons (or positrons) is not a suitable situation.

On the other hand, one gets into the desired regime much easier with heavier particles, of which the best studied is channeling of positive and negative muons. If a muon travels with velocity of an atomic scale Ze^2/\hbar , so that the coupling (98) is of the order of one, its momentum is very large:

$$p_\mu \simeq \frac{m_\mu}{m_e} \frac{\hbar}{a_B} \gg p_{atomic} = \frac{\hbar}{a_B} \quad (99)$$

Correspondingly, the particle wavelength will be very small compared to the core potential radius a_B , and thus the problem is semiclassical and our results are applicable.

The theory we described above predicts delocalization in momentum space. In terms of waves scattering it means that a plane wave, after multiple scattering on the core potentials, will give rise to many other plane waves, and eventually there will be almost no memory preserved about the initial plane wave. Following the discussion of Sec. IV, the number of plane waves which are in a resonance with a given plane wave can be estimated as

$$\lambda n \ln(p_\mu/p_{atomic}) \gg 1 \quad (100)$$

At large p_μ/p_{atomic} one can have many resonances between plane waves even for a weak coupling situation, $\lambda n \leq 1$. In this case, the theory presented in this paper will be accurate.

There are several simple predictions which can be tested experimentally. It follows from our discussion of resonances that the number of plane waves into which an initial plane wave is scattered, although large, is much smaller than the total number of plane waves with energies close to the initial energy. The origin of this effect is in the critical nature of delocalized states for a Coulomb potential. The diffusion over a constant energy surface in momentum space leads to a nontrivial angular correlation function, with a power law singularity (91) at small angles.

X. GENERAL PICTURE

Our discussion of the localization in momentum space focused on the Coulomb scatterers which have a marginal singularity and give rise to critical states in momentum space. It is easy to see how the RG results will change for a general power law singularity, $U(r) = \lambda/r^\sigma$. The hopping in momentum space, given by the Fourier transform of $U(r)$, will be more short-range if $\sigma < 1$ and less short-range if $\sigma > 1$. Hence, at $\sigma > 1$ the system is always in the delocalized phase, no matter what λ . On the other hand, for $\sigma < 1$ the RG will scale coupling down to zero, and we can conclude that for sufficiently small λ (i.e., when the RG is applicable) all states are localized.

To represent this graphically, one can draw a schematic phase diagram in the plane (σ, λ) (see Fig. X). The region marked by **I** corresponds to the standard delocalization transition for short-range hopping (large σ). The region marked by **II** corresponds to the transition at small λ studied in this article. One can note that the critical behavior in the cases **I** and **II** is quite different: all scaling relations in the case **II** depend on the dimensionless coupling strength which does not change under the RG flow, whereas in the case **I** there are no parameters. This means that one cannot go continuously from one scaling regime to the other, and hence we conjecture that there must be a point on the localization-delocalization phase boundary at which the critical behavior changes.

Another reason to believe that there is such a point is that in the RG theory presented above the transition occurs at the same $\sigma = 1$ for any λ (which must be sufficiently small). So the phase boundary in Fig. X has a part that goes straight up. This suggests that the

special point at which the critical behavior changes must simultaneously be the point **C** where the phase boundary turns away from the $\sigma = 1$ straight line.

One more reason supporting this phase diagram follows from recent paper by Mirlin and Fyodorov (see Ref. [16]). They studied a one-dimensional problem with power-law hopping in the regime of strong hopping where one can use the supersymmetric sigma-model. In the (σ, λ) plane this corresponds to large λ . In this region, a delocalization transition was found at certain critical σ . It is natural to make a connection with our results by assuming that the point **C** is a tricritical point at which all phase boundaries match.

To summarize, in this article we consider the problem of Bloch states for a fast particle moving in a crystal of scatterers with power law singularity. This problem is shown to be equivalent to a localization problem in momentum space which we study using an RG previously developed for the localization with long-range hopping. We find a localization transition with the critical singularity being that of Coulomb potential. In the localized phase the particle state is close to a single plane wave, whereas in the delocalized phase the state is a superposition of many plane waves with wavevectors spread uniformly over a constant energy shell in momentum space. In other words, the localization transition is the transition between integrable and ergodic dynamics. We believe that the relation between delocalization in momentum space and the transition to chaos is very generic, and will be useful beyond the problem of power-law scatterers. One can hypothesize that understanding of Quantum Chaos in general is in some way a problem of finding an equivalent in localization theory.

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FIGURES

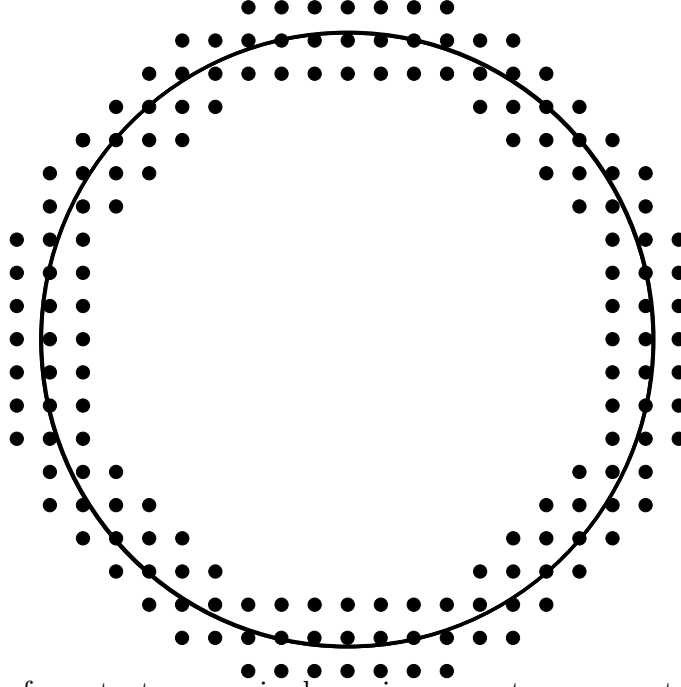


FIG. 1. A sphere of constant energy is shown in momentum space, together with a shell of the dual crystal lattice sites (7) which are close to the sphere. Delocalization of Bloch states in momentum space occurs within this shell. Effective dimension of the space in which we consider the localization problem is given by the sphere dimension, i.e., equals two.

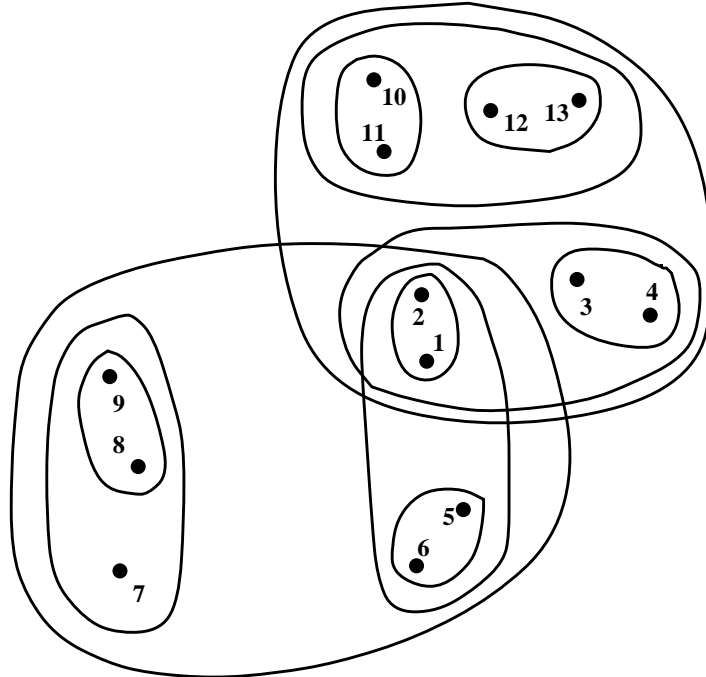


FIG. 2. Hierarchical structure of two eigenstates is shown: *a*) $((1+2)+(6+5))+((8+9)+7)$; *b*) $((1+2)+(3+4))+((10+11)+(12+13))$; where we denote by $(n+m)$ a resonance pair formed by the states m and n . The time hierarchy of excitation transport is given by the same diagram.

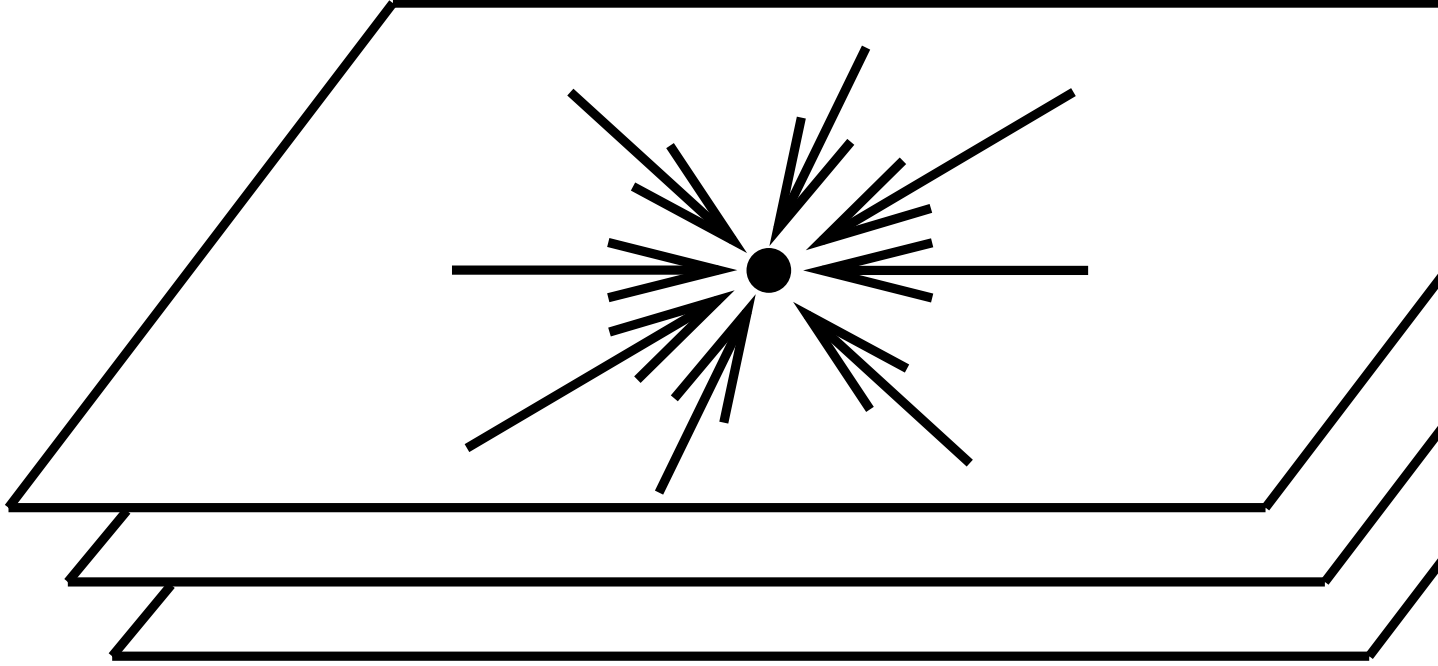


FIG. 3. Schematic view of the RG flow given by Eq. (44). The space of functions $f(a)$ is represented by a bundle of invariant surfaces labeled by the integrals $G_{\alpha\beta} = \langle \bar{a}_\alpha a_\beta \rangle$, ($\alpha, \beta = 1, 2$, $\alpha \leq \beta$). Restricted on each of the surfaces the system has one attracting fix point (given by Expr. (55)).

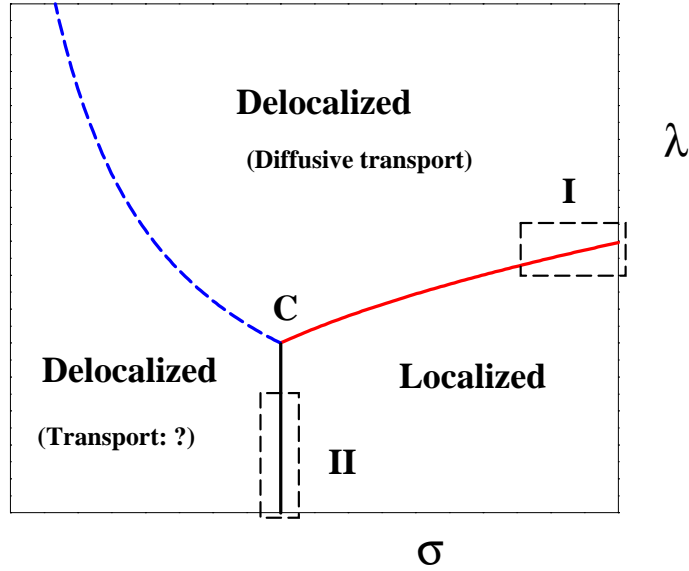


FIG. 4. Phase diagram in the $(\sigma - \lambda)$ plane. (Both σ and λ vary between 0 and ∞ .) The phase boundaries are marked corresponding to the delocalization by long-range forces studied in this work (region *II*) and to the conventional Anderson transition at short-range hopping (region *I*).